

Transverse Ward-Takahashi Identity, Anomaly and Schwinger-Dyson Equation *

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Abstract

Based on the path integral formalism, we rederive and extend the transverse Ward-Takahashi identities (which were first derived by Yasushi Takahashi) for the vector and the axial vector currents and simultaneously discuss the possible anomaly for them. Subsequently, we propose a new scheme for writing down and solving the Schwinger-Dyson equation in which the the transverse Ward-Takahashi identity together with the usual (longitudinal) Ward-Takahashi identity are applied to specify the fermion-boson vertex function. Especially, in two dimensional Abelian gauge theory, we show that this scheme leads to the exact and closed Schwinger-Dyson equation for the fermion propagator in the chiral limit (when the bare fermion mass is zero) and that the Schwinger-Dyson equation can be exactly solved.

Key words: Ward-Takahashi identity, anomaly, Schwinger-Dyson equation, chiral symmetry, exact solution

PACS numbers:

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1 Introduction

If a quantum field theory possesses some symmetry, there exist various identities among Green functions of the theory. They are in general called the Ward-Takahashi (WT) identities [1]. If the action is invariant under the continuous symmetry (i.e. transformation with continuous parameters) of the dynamical variable, this inevitably leads to the existence of the corresponding conservation law, as Noether's theorem claims. However, the symmetry in the classical theory (at tree level) may be broken in the quantum theory (at loop level), which is the phenomenon known as the anomaly [2]. If we could construct all the correlation functions (more precisely, the Wightman function) satisfying all the (infinite number of) WT identities associated with some symmetry, we would be in principle able to recover the original theory with the symmetry in question.

For recent two decades, this kind of attempt has been extensively performed in the framework of the Schwinger-Dyson (SD) equation [3] for the purpose of studying the strong coupling phase of the gauge theory [4, 5, 6]. The strong coupling phase of QED is believed to exist above the critical coupling e_c of order unity, which is the phase transition point accompanied by the spontaneous breakdown of chiral symmetry [7]. It is notorious that the simple approximation (called ladder or rainbow approximation) leads to the severely gauge-dependent result for the gauge-invariant quantity. This is because we must truncate the infinite series of SD equations so that we are able to handle with them in actually solving the SD equation and the appropriate procedure of truncating the series in a gauge-invariant manner is not known, although, in the perturbation theory, the gauge-invariance is preserved order by order by using the gauge-invariant regularization. In light of this, the fermion-boson (photon) vertex function $\Gamma_\mu(x, y; z)$ is the most difficult quantity to be specified as we first encounter in the framework of the SD equation of gauge theory. In recent several years, however, there have been considerable efforts to improve the vertex so that it satisfies the WT identity as a result of gauge invariance, although there are infinite number of WT identities coming from the gauge invariance. The longitudinal part of Γ_μ can be written in terms of the full fermion propagator S , that is to say, in momentum space

$$k_\mu \Gamma^\mu(q, p) = S^{-1}(q) - S^{-1}(p), \quad k_\mu := q_\mu - p_\mu. \quad (1.1)$$

This observation might suggest a possibility of writing down the self-consistent and closed SD equation for the full fermion propagator. Recently it has been fully recognized that the usual WT identity is not sufficient to specify the vertex function in the non-perturbative study, actually it does not at all specify the transverse part of the vertex. Some people have tried to determine or choose the most plausible form for the vertex by requiring a number of conditions: no kinematic singularity, multiplicative renormalizability, agreement with the lower order perturbation theory, etc. [8, 9, 10]. Such a sort of attempt reached to the top quite recently at least in the quenched case (i.e. the limit of zero fermion-flavor or no vacuum polarization to the photon propagator) [11]. The SD equation for the (3-point) vector vertex function is not closed. This is one of the reasons why the gauge theory defined on space-time in dimensions greater than two can not be solved exactly. Moreover, the vector vertex

function may have various types of tensor structures [12], in sharp contrast with the full fermion and the full photon propagators (2-point functions). This fact makes the actual analysis more difficult.

An aim of this paper lies in providing more information which is useful to specify the transverse vertex based on a new type of WT identity [13], which is called the transverse WT identity. In contrast, the usual WT identity is called the longitudinal WT identity. The transverse WT identity was derived by Takahashi more than ten years ago [13]. In this paper, we rederive it from the path integral formalism and examine the possibility of the anomaly associated with it. The transverse WT identity specifies the *rotation* of the vertex

$$\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu, \quad (1.2)$$

while the longitudinal WT identity determines its divergence

$$\partial^\mu \Gamma_\mu. \quad (1.3)$$

We point out that two types of combination (1.2) and (1.3) are enough to specify the vertex uniquely in the SD equation for the fermion propagator, although they do not specify the vertex function itself. It should be kept in mind that the transverse WT identity is not closed in itself, since the rotation can not in general be written only in terms of the fermion propagator. In this point, the transverse part is quite different from the longitudinal part. Nevertheless, we show that, in two dimensional Abelian gauge theory, the transverse WT identity gives the closed set of SD equations (in the limit of zero bare fermion mass) together with the two propagators, and a set of SD equation can be exactly soluble. In $D > 2$ dimensions, such a simple situation does not occur. But the truncated transverse WT identity in the same form as the two-dimensional case leads to the exactly soluble SD equation for the fermion propagator in $D > 2$ dimensions without any further approximation (linearization and separation of the kernel for the angular variable). This may be an alternative starting point in looking for the appropriate ansatz for the vertex.

This paper is organized as follows. As a preliminary to the subsequent sections, in section 2 we review the usual WT identity for fixing the notation. In section 3, we rederive the transverse WT identity based on the path integral formalism. Our presentation allows straightforward extension to the non-Abelian gauge theory. In section 4, the transverse WT identity for the axial vector current is derived in the same way as in section 3. In these two sections, 3 and 4, we neglect the additional term which may come from the anomaly. The anomaly is taken into account in section 5 based on the Fujikawa's method [14] in which the anomaly is identified with the Jacobian accompanied by the transformation of fermionic variables in the path integral measure. This gives an alternative derivation of the transverse WT identity as well as the longitudinal WT identity. In section 6, we apply the transverse WT identity to the SD equation for the fermion propagator. We propose a new strategy to specify the SD equation. In section 7, we show that in 1+1 dimensions the transverse and the longitudinal WT identities can lead to the exact and closed SD equation and the SD equation derived in such a way can be exactly solvable. In the final section we summarize the result.

2 Usual longitudinal WT identity

We consider the theory with the Lagrangian

$$\begin{aligned}\mathcal{L}[\bar{\Psi}, \Psi, A] &= \mathcal{L}_F[\bar{\Psi}, \Psi, A] + \mathcal{L}_g[A], \\ \mathcal{L}_F[\bar{\Psi}, \Psi, A] &:= \bar{\Psi} i \gamma^\mu (\partial_\mu - i e A_\mu) \Psi - \bar{\Psi} M \Psi,\end{aligned}\tag{2.1}$$

where \mathcal{L}_g is the Lagrangian for the gauge field specified below, Ψ is the Dirac fermion with a spinor index α and a flavor index i (and possibly a color index for non-Abelian gauge theory) and M is a mass matrix for the fermion. For a while, we restrict our attention to the Abelian gauge case and define the vector current \mathcal{J}_μ by ¹

$$\mathcal{J}_\mu(x) := \bar{\Psi}(x) \gamma_\mu \Psi(x) = \bar{\Psi}_\alpha^i(x) (\gamma_\mu)^{\alpha\beta} \Psi_\beta^i(x).\tag{2.2}$$

It is well known that the usual WT identity is derived from

$$\mathcal{D}(x)_J := \partial_\mu \langle \mathcal{J}^\mu(x) \rangle_J = i \langle \bar{\Psi}(x) \eta(x) \rangle_J - i \langle \bar{\eta}(x) \Psi(x) \rangle_J,\tag{2.3}$$

where we have defined the expectation value $\langle \dots \rangle_J$ in the presence of a set of external sources $J := \{J^\mu, \eta, \bar{\eta}\}$ by using the functional integral:

$$\langle \mathcal{O}(x) \rangle_J = \frac{[[\mathcal{O}(x)]]_J}{[[1]]_J},\tag{2.4}$$

$$[[\mathcal{O}(x)]]_J = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \mathcal{D}A_\mu \exp \left[i \int d^D x (\mathcal{L} + \mathcal{L}_J) \right] \mathcal{O}(x),\tag{2.5}$$

with Schwinger's source term:

$$\mathcal{L}_J := A_\mu(x) J^\mu(x) + \bar{\Psi}(x) \eta(x) + \bar{\eta}(x) \Psi(x).\tag{2.6}$$

Here note that the external sources have possible indices corresponding to those of the field.

On the other hand, if we consider the Abelian gauge theory whose Lagrangian (with a gauge-fixing term) is given by

$$\mathcal{L}_g[A] = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial^\mu A_\mu)^2,\tag{2.7}$$

the identity

$$\int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \mathcal{D}A_\mu \frac{\delta}{\delta A_\mu} \exp \left[i \int d^D x (\mathcal{L} + \mathcal{L}_J) \right] \equiv 0,\tag{2.8}$$

leads to

$$\langle \mathcal{J}^\mu(x) \rangle_J = -\frac{1}{e} \langle \Delta^{\mu\rho}(\partial) A_\rho(x) + J^\mu \rangle_J,\tag{2.9}$$

¹In the following we do not write the indices explicitly when the indices are contracted in a obvious way. In this paper $A := B$ implies that A is defined by B .

where $\Delta_{\mu\rho}(\partial)$ is the inverse of the free boson propagator $D_{\mu\rho}^{(0)}(\partial)$:

$$\Delta_{\mu\rho}(\partial) := D_{\mu\rho}^{(0)-1}(\partial) = \partial^2 g_{\mu\rho} - \partial_\mu \partial_\rho + \xi^{-1} \partial_\mu \partial_\rho. \quad (2.10)$$

Now we introduce the generating functional for the connected correlation function:

$$iW(J^\mu, \eta, \bar{\eta}) := \ln Z(J^\mu, \eta, \bar{\eta}), \quad Z(J^\mu, \eta, \bar{\eta}) = [[1]]_J. \quad (2.11)$$

Then Eq. (2.9) reads

$$\langle \mathcal{J}^\mu(x) \rangle_J = -\frac{1}{e} \left[\Delta^{\mu\rho}(\partial) \frac{\delta W}{\delta J^\rho} + J^\mu \right], \quad (2.12)$$

since

$$\frac{\delta W}{\delta J^\mu(x)} = \langle A_\mu(x) \rangle_J, \quad \frac{\delta W}{\delta \eta(x)} = -\langle \bar{\Psi}(x) \rangle_J, \quad \frac{\delta W}{\delta \bar{\eta}(x)} = \langle \Psi(x) \rangle_J. \quad (2.13)$$

From Eq. (2.12), we obtain

$$\begin{aligned} \partial_\mu \langle \mathcal{J}^\mu(x) \rangle_J &= -\frac{1}{e} \left[\partial_\mu \Delta^{\mu\rho}(\partial) \frac{\delta W}{\delta J^\rho} + \partial_\mu J^\mu(x) \right] \\ &= -\frac{1}{e} \langle \partial_\mu \Delta^{\mu\rho}(\partial) A_\rho(x) + \partial_\mu J^\mu \rangle_J, \end{aligned} \quad (2.14)$$

where only the longitudinal part of the (inverse) boson propagator contributes to the right-hand-side of this equation. Note that Eq. (2.3) is rewritten as

$$\partial_\mu \langle \mathcal{J}^\mu(x) \rangle_J = -i \frac{\delta W}{\delta \eta(x)} \eta(x) - i \bar{\eta}(x) \frac{\delta W}{\delta \bar{\eta}(x)}. \quad (2.15)$$

Then, differentiating Eq. (2.14) and Eq. (2.15) with respect to $\bar{\eta}(y)$ and $\eta(z)$ and then putting $J^\mu = \eta = \bar{\eta} = 0$, we obtain

$$\begin{aligned} \mathcal{D}(x, y, z) &:= \partial_\mu \langle \mathcal{J}^\mu(x); \Psi(y); \bar{\Psi}(z) \rangle_c \\ &= \langle \frac{1}{e} \partial_\mu \Delta^{\mu\rho}(\partial) A_\rho(x); \Psi(y); \bar{\Psi}(z) \rangle_c \\ &= \langle \Psi(y) \bar{\Psi}(z) \rangle_c \delta^D(x - z) - \langle \Psi(y) \bar{\Psi}(z) \rangle_c \delta^D(x - y), \end{aligned} \quad (2.16)$$

where $\langle \dots \rangle_c$ denotes the connected correlation function. For the proper fermion-boson vertex function in momentum representation

$$S(q) \Gamma^\mu(q, p) S(p) := \int d^D y \int d^D z e^{i(q \cdot y - p \cdot z)} \langle \mathcal{J}^\mu(0); \Psi(y); \bar{\Psi}(z) \rangle_c, \quad (2.17)$$

the well-known form of the WT identity is recovered:

$$k_\mu \Gamma^\mu(q, p) = S^{-1}(q) - S^{-1}(p), \quad k_\mu := q_\mu - p_\mu. \quad (2.18)$$

The full gauge-boson propagator $D_{\mu\nu}(\partial)$ obeys the SD equation:

$$D_{\mu\nu}^{-1}(\partial) = D_{\mu\nu}^{(0)-1}(\partial) - \Pi_{\mu\nu}(\partial). \quad (2.19)$$

In gauge theory, the longitudinal part of the full gauge-boson propagator is the same as that of the free one, since

$$\Pi_{\mu\nu}(\partial) = (g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\Pi(\partial), \quad (2.20)$$

due to gauge invariance. Therefore, we obtain

$$\partial^\mu D_{\mu\rho}^{(0)-1}(\partial) = \xi^{-1}\partial^2\partial_\rho = \partial^\mu D_{\mu\rho}^{-1}(\partial). \quad (2.21)$$

Hence this property can be also derived as a consequence of the WT identity.²

Furthermore, we introduce the generating functional for the one-particle irreducible diagram through the Legendre transform

$$\Gamma(\bar{\Psi}, \Psi, A_\mu) := W(\eta, \bar{\eta}, J^\mu) - \int d^D x [A_\mu(x)J^\mu(x) + \bar{\Psi}(x)\eta(x) + \bar{\eta}(x)\Psi(x)]. \quad (2.22)$$

By making use of the relations following from (2.22):

$$\frac{\delta\Gamma}{\delta A_\mu(x)} = -J^\mu(x), \quad \frac{\delta\Gamma}{\delta \bar{\Psi}(x)} = -\eta(x), \quad \frac{\delta\Gamma}{\delta \Psi(x)} = \bar{\eta}(x), \quad (2.23)$$

Eq. (2.9) is further rewritten as

$$\langle \mathcal{J}^\mu(x) \rangle = -\frac{1}{e} \left[\Delta^{\mu\rho}(\partial) A_\rho(x)_J - \frac{\delta\Gamma}{\delta A^\mu(x)} \right], \quad (2.24)$$

while Eq. (2.3) is rewritten as

$$\partial_\mu \langle \mathcal{J}^\mu(x) \rangle_J = -i\bar{\Psi}(x) \frac{\delta\Gamma}{\delta \bar{\Psi}(x)} - i \frac{\delta\Gamma}{\delta \Psi(x)} \Psi(x). \quad (2.25)$$

Substituting Eq. (2.24) into Eq. (2.25), we obtain

$$-\mathcal{D}(x)_J = \frac{1}{e} \left[\partial_\mu \Delta^{\mu\rho}(\partial) A_\rho(x) - \partial_\mu \frac{\delta\Gamma}{\delta A^\mu(x)} \right] = i\bar{\Psi}(x) \frac{\delta\Gamma}{\delta \bar{\Psi}(x)} + i \frac{\delta\Gamma}{\delta \Psi(x)} \Psi(x). \quad (2.26)$$

The WT identity for the fermion-boson vertex function is obtained by taking derivatives of both side of Eq. (2.26) in $\bar{\Psi}(z)$ and $\Psi(y)$ and finally setting all the classical fields equal to zero, $J^\mu = \eta = \bar{\eta} = 0$:

$$\frac{1}{e} \partial_\mu^x \frac{\delta^3\Gamma}{\delta A^\mu(x) \delta \Psi(y) \delta \bar{\Psi}(z)} + i \frac{\delta^2\Gamma}{\delta \Psi(y) \delta \bar{\Psi}(z)} \delta(x-z) - i \frac{\delta^2\Gamma}{\delta \Psi(y) \delta \bar{\Psi}(z)} \delta(x-y) = 0. \quad (2.27)$$

² If the vector boson has the mass μ , the inverse of the free propagator reads $D_{\mu\rho}^{(0)-1}(\partial) = \partial^2 g_{\mu\rho} - \partial_\mu\partial_\rho + \xi^{-1}\partial_\mu\partial_\rho + \mu^2 g_{\mu\rho}$. Usually, the naive introduction of such a term breaks the gauge invariance and hence it seems at first glance that this property can not be preserved. However, if we introduce the additional scalar freedom (Stückelberg field) and adopt the R_ξ gauge, we can extend this scheme into the massive gauge-boson model. This issue will be discussed in detail in a subsequent paper [21].

From the definitions:

$$\Gamma_\mu(x, y, z) := \frac{\delta^3 \Gamma}{\delta A^\mu(x) \delta \Psi(y) \delta \bar{\Psi}(z)}, \quad (2.28)$$

and

$$\frac{\delta^2 \Gamma}{\delta \Psi(y) \delta \bar{\Psi}(z)} = - \left[\frac{\delta^2 W}{\delta \eta(y) \delta \bar{\eta}(z)} \right]^{-1} = \langle \Psi(y) \bar{\Psi}(z) \rangle^{-1}, \quad (2.29)$$

we can see that (2.27) reproduces (2.18) in momentum representation, using the Fourier transformation:

$$(2\pi)^D \delta^D(k - p + q) \Gamma_\mu(q, p) := \int d^D x d^D y d^D z e^{i(q \cdot y - p \cdot z - k \cdot x)} \Gamma_\mu(x, y, z). \quad (2.30)$$

The form (2.27) for the WT identity is known to be very useful in the proof of renormalizability, which is not the subject of this paper.

3 Derivation of the transverse WT identity

The usual WT identity for the vector vertex has been obtained by taking the divergence of the current expectation value, Eq. (2.14). Furthermore, we want to find the relation which specifies the rotation of the current expectation value:

$$\mathcal{R}_{\mu\nu}(x)_J := \partial_\mu \langle \mathcal{J}_\nu(x) \rangle_J - \partial_\nu \langle \mathcal{J}_\mu(x) \rangle_J. \quad (3.1)$$

First of all, we find from Eq. (2.12) and Eq. (2.24)

$$\begin{aligned} \mathcal{R}_{\mu\nu}(x)_J &= -\frac{1}{e} [\partial_\nu \Delta_{\mu\rho}(\partial) - \partial_\mu \Delta_{\nu\rho}(\partial)] A_\rho(x) - \frac{1}{e} [\partial_\nu J_\mu - \partial_\mu J_\nu] \\ &= -\frac{1}{e} \partial^2 \left[\partial_\mu \frac{\delta W}{\delta J^\nu} - \partial_\nu \frac{\delta W}{\delta J^\mu} \right] - \frac{1}{e} [\partial_\nu J_\mu - \partial_\mu J_\nu] \\ &= -\frac{1}{e} \partial^2 [\partial_\mu A_\nu(x)_J - \partial_\nu A_\mu(x)_J] - \frac{1}{e} \left[\partial_\nu \frac{\delta \Gamma}{\delta A^\mu(x)} - \partial_\mu \frac{\delta \Gamma}{\delta A^\nu(x)} \right], \end{aligned} \quad (3.2)$$

where we have used ³

$$\partial_\nu \Delta_{\mu\rho}(\partial) - \partial_\mu \Delta_{\nu\rho}(\partial) = \partial^2 (\partial_\mu g_{\nu\rho} - \partial_\nu g_{\mu\rho}). \quad (3.3)$$

Next, we need to know another expression for the rotation $\mathcal{R}_{\mu\nu}^J$. In order to derive such relations, it turns out that we have only to pay attention to the fermionic part

$$\mathcal{L}_F = \bar{\Psi} i \gamma^\mu (\partial_\mu - ie A_\mu) \Psi - \bar{\Psi} M \Psi + \bar{\eta} \Psi + \bar{\Psi} \eta. \quad (3.4)$$

³ In contrast to the longitudinal part, the transverse part does not in general have the same form as the free case.

Therefore the following relations hold also for the non-Abelian case irrespective of the gauge part \mathcal{L}_g , if we identify A_μ in \mathcal{L}_F as $A_\mu(x) = A_\mu^a(x)T^a$ ($a = 1, \dots, \dim G$) with the generator T^a of the gauge group G .

The identity

$$\int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \mathcal{D}A_\mu \frac{\delta}{\delta \bar{\Psi}(x)} \exp \left[i \int d^D x (\mathcal{L} + \mathcal{L}_J) \right] \equiv 0, \quad (3.5)$$

leads to

$$[[i\gamma^\mu [\partial_\mu - ieA_\mu(x)]\Psi(x) - M\Psi(x) + \eta(x)]]_J = 0, \quad (3.6)$$

where we have omitted to write the indices explicitly. We multiply Eq.(3.6) by the matrix Γ from the left where Γ may have spinor, flavor and color indices. And then, operating the differential operator $\frac{\delta}{\delta \eta(y)}$ to the resulting equation, we obtain

$$\begin{aligned} & \langle -\bar{\Psi}(y) i\gamma^\mu [\partial_\mu - ieA_\mu(x)]\Psi(x) + \bar{\Psi}(y) \Gamma M \Psi(x) - \bar{\Psi}(y) \Gamma \eta(x) \rangle_J \\ &= -\text{tr}(\Gamma) \delta(x - y), \end{aligned} \quad (3.7)$$

where we should be careful with the anticommuting nature of the Grassmann variable and the ordering of spinor, flavor and color indices, and the trace is taken over all the indices.

On the other hand, the identity

$$\int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \mathcal{D}A_\mu \frac{\delta}{\delta \bar{\Psi}(x)} \exp \left[i \int d^D x (\mathcal{L} + \mathcal{L}_J) \right] \equiv 0, \quad (3.8)$$

leads to

$$[[\bar{\Psi}(x) i\gamma^\mu [\overleftarrow{\partial}_\mu + ieA_\mu(x)] + m\bar{\Psi}(x) - \bar{\eta}(x)]]_J = 0. \quad (3.9)$$

Similarly, multiplying Eq. (3.9) by the same matrix Γ as above from the right and subsequently operating $\frac{\delta}{\delta \bar{\eta}(y)}$, we obtain

$$\begin{aligned} & \langle -\bar{\Psi}(x) i[\overleftarrow{\partial}_\mu + ieA_\mu(x)]\gamma^\mu \Gamma \Psi(y) - \bar{\Psi}(x) \Gamma M \Psi(y) + \bar{\eta}(x) \Gamma \Psi(y) \rangle_J \\ &= \text{tr}(\Gamma) \delta^D(x - y). \end{aligned} \quad (3.10)$$

By adding Eq. (3.10) to Eq. (3.7) or subtracting Eq. (3.10) from Eq. (3.7), and subsequently setting $x = y$, i.e. $\left[\frac{\delta}{\delta \eta(y)} [\Gamma \times (3.6)] \pm \frac{\delta}{\delta \bar{\eta}(y)} [(3.9) \times \Gamma] \right] \Big|_{x=y}$, we get two sets of WT identities:

$$\begin{aligned} \partial_\rho \langle \bar{\Psi} \frac{i}{2} \{ \Gamma, \gamma^\rho \} \Psi \rangle_J &= \langle \bar{\Psi} [\Gamma, M] \Psi \rangle_J - \langle \bar{\Psi} \Gamma \eta \rangle_J + \langle \bar{\eta} \Gamma \Psi \rangle_J \\ &\quad - \langle \bar{\Psi} \frac{i}{2} [\Gamma, \gamma^\rho] (\overrightarrow{\partial}_\rho - \overleftarrow{\partial}_\rho) \Psi \rangle_J + ie \langle \bar{\Psi} i [\Gamma, \gamma^\rho A_\rho] \Psi \rangle_J \\ &\quad + \mathcal{A}_\Gamma[A_\mu], \end{aligned} \quad (3.11)$$

$$\begin{aligned}
\partial_\rho \langle \bar{\Psi} \frac{i}{2} [\Gamma, \gamma^\rho] \Psi \rangle_J &= \langle \bar{\Psi} \{ \Gamma, M \} \Psi \rangle_J - \langle \bar{\Psi} \Gamma \eta \rangle_J - \langle \bar{\eta} \Gamma \Psi \rangle_J \\
&\quad - \langle \bar{\Psi} \frac{i}{2} \{ \Gamma, \gamma^\rho \} (\vec{\partial}_\rho - \overleftarrow{\partial}_\rho) \Psi \rangle_J + ie \langle \bar{\Psi} i \{ \Gamma, \gamma^\rho A_\rho \} \Psi \rangle_J \\
&\quad + 2\text{tr}(\Gamma) \delta^D(0) + \mathcal{A}_\Gamma[A_\mu],
\end{aligned} \tag{3.12}$$

where we have introduced the commutation relation $[A, B] := AB - BA$ and the anticommutation relation $\{A, B\} := AB + BA$. Here $\mathcal{A}_\Gamma[A_\mu]$ denotes the possible anomaly which will be discussed in section 5.

In what follows, we choose Γ to be a direct product of the each factor for spinor, flavor and color indices: $\Gamma = \Gamma_S \otimes \Gamma_F \otimes \Gamma_C$. It turns out that the usual WT identity Eq. (2.3) for the Abelian vector current is obtained from Eq. (3.11) by choosing

$$\Gamma = 1_S \otimes 1_F \otimes 1_C. \tag{3.13}$$

A new type of WT identities, so-called the transverse WT identity [13], is obtained from Eq. (3.12) by choosing

$$\Gamma = \sigma_{\mu\nu} \otimes 1_F \otimes 1_C, \quad \sigma_{\mu\nu} := \frac{i}{2} [\gamma_\mu, \gamma_\nu]. \tag{3.14}$$

Indeed, this choice of Γ yields

$$\partial_\rho \langle \bar{\Psi}(x) \frac{i}{2} [\Gamma, \gamma^\rho] \Psi(x) \rangle_J = \partial_\mu \langle \bar{\Psi}(x) \gamma_\nu \Psi(x) \rangle_J - \partial_\nu \langle \bar{\Psi}(x) \gamma_\mu \Psi(x) \rangle_J. \tag{3.15}$$

Here it should be remarked that

$$\sigma_{\mu\nu} \gamma_\rho = \frac{1}{2} [\sigma_{\mu\nu}, \gamma_\rho] + \frac{1}{2} \{ \sigma_{\mu\nu}, \gamma_\rho \} = i(\gamma_\mu g_{\nu\rho} - \gamma_\nu g_{\mu\rho}) + \frac{1}{2} \{ \sigma_{\mu\nu}, \gamma_\rho \}, \tag{3.16}$$

which follows from $\{ \gamma_\mu, \gamma_\nu \} = 2g_{\mu\nu}$. (As we will see shortly, the symmetrized part $\{ \sigma_{\mu\nu}, \gamma_\rho \}$ changes depending on the space-time dimension.) Thus we arrive at the desired expression (up to anomaly):

$$\begin{aligned}
\mathcal{R}_{\mu\nu}(x)_J &= \langle \bar{\Psi}(x) \{ \sigma_{\mu\nu}, M \} \Psi(x) \rangle_J - \langle \bar{\Psi}(x) \sigma_{\mu\nu} \eta(x) \rangle_J - \langle \bar{\eta}(x) \sigma_{\mu\nu} \Psi(x) \rangle_J \\
&\quad - \langle \bar{\Psi}(x) \frac{i}{2} \{ \sigma_{\mu\nu}, \gamma^\rho \} (\vec{\partial}_\rho - \overleftarrow{\partial}_\rho) \Psi(x) \rangle_J \\
&\quad - e \langle \bar{\Psi}(x) \{ \sigma_{\mu\nu}, \gamma_\rho \} \Psi(x) A^\rho(x) \rangle_J.
\end{aligned} \tag{3.17}$$

Taking derivatives of both side of Eq. (3.17) with respect to $\bar{\Psi}(z)$ and $\Psi(y)$ and setting all the sources equal to zero, $J^\mu = \eta = \bar{\eta} = 0$, we get the transverse WT identity [13]:

$$\begin{aligned}
\mathcal{R}_{\mu\nu}(x, y, z) &:= \partial_\mu \langle \bar{\Psi}(x) \gamma_\nu \Psi(x); \Psi(y) \bar{\Psi}(z) \rangle_c - \partial_\nu \langle \bar{\Psi}(x) \gamma_\mu \Psi(x); \Psi(y) \bar{\Psi}(z) \rangle_c \\
&= \langle \bar{\Psi}(x) \{ \sigma_{\mu\nu}, M \} \Psi(x); \Psi(y) \bar{\Psi}(z) \rangle_c \\
&\quad - \langle \Psi(y) \bar{\Psi}(x) \rangle_c \sigma_{\mu\nu} \delta^D(x - z) - \sigma_{\mu\nu} \langle \Psi(x) \bar{\Psi}(z) \rangle_c \delta^D(x - y) \\
&\quad - \langle \bar{\Psi}(x) \frac{i}{2} \{ \sigma_{\mu\nu}, \gamma^\rho \} (\vec{\partial}_\rho - \overleftarrow{\partial}_\rho) \Psi(x); \Psi(y) \bar{\Psi}(z) \rangle_c \\
&\quad - e \langle \bar{\Psi}(x) \{ \sigma_{\mu\nu}, \gamma_\rho \} \Psi(x) A^\rho(x); \Psi(y) \bar{\Psi}(z) \rangle_c.
\end{aligned} \tag{3.18}$$

The transverse WT identities were first found by Takahashi based on the canonical formalism in the 3+1 dimension where

$$\frac{1}{2}\{\sigma_{\mu\nu}, \gamma_\rho\} = \epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma^\sigma. \quad (3.19)$$

In this paper we rederived them based on the path integral formalism. We find that the transverse WT identity exhibits different appearance depending on the dimensionality of space-time.

In 2+1 dimensions, we choose the gamma matrices as

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_2. \quad (3.20)$$

Then $\sigma_{\mu\nu} = -\epsilon_{\mu\nu\sigma}\gamma^\sigma$ ($\epsilon_{012} = 1$) and hence

$$\frac{1}{2}\{\sigma_{\mu\nu}, \gamma_\rho\} = -\epsilon_{\mu\nu\rho}. \quad (3.21)$$

And in 1+1 dimensional space-time, we choose

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1, \quad \gamma^5 := \gamma^0\gamma^1 = \sigma_3, \quad (\epsilon_{01} = 1), \quad (3.22)$$

which implies

$$\sigma_{\mu\nu} = i\epsilon_{\mu\nu}\gamma_5, \quad \gamma_\mu\gamma_5 = \epsilon_{\mu\nu}\gamma^\nu. \quad (3.23)$$

Therefore, we obtain

$$\{\sigma_{\mu\nu}, \gamma_\rho\} = i\epsilon_{\mu\nu}\{\gamma_5, \gamma_\rho\} \equiv 0. \quad (3.24)$$

In 1+1 dimensions, therefore, Eq. (3.17) and Eq. (3.18) have remarkably simple forms:

$$\mathcal{R}_{\mu\nu}(x)_J = \langle \bar{\Psi}(x)\{\sigma_{\mu\nu}, M\}\Psi(x) \rangle_J - \langle \bar{\Psi}(x)\sigma_{\mu\nu}\eta(x) \rangle_J - \langle \bar{\eta}(x)\sigma_{\mu\nu}\Psi(x) \rangle_J, \quad (3.25)$$

$$\begin{aligned} \mathcal{R}_{\mu\nu}(x, y, z) &= \langle \bar{\Psi}(x)\{\sigma_{\mu\nu}, M\}\Psi(x); \Psi(y)\bar{\Psi}(z) \rangle_c \\ &\quad - \langle \Psi(y)\bar{\Psi}(x) \rangle_c \sigma_{\mu\nu} \delta^D(x - z) - \sigma_{\mu\nu} \langle \Psi(x)\bar{\Psi}(z) \rangle_c \delta^D(x - y) \end{aligned} \quad (3.26)$$

In the chiral limit $M = 0$, especially, the transverse WT identity leads to the surprisingly simple identity for the rotation of the vector vertex. For example, in QED₂, we obtain by making use of Eq. (2.17):

$$k_\mu \Gamma_\nu(q, p) - k_\nu \Gamma_\mu(q, p) = S^{-1}(q)\sigma_{\mu\nu} + \sigma_{\mu\nu}S^{-1}(p). \quad (3.27)$$

The non-Abelian versions of the transverse WT identities as well as the usual longitudinal WT identities for the current

$$\mathcal{J}_\mu^a(x) := \bar{\Psi}(x)\gamma_\mu T^a\Psi(x), \quad (3.28)$$

are obtained from (3.11) and (3.12), if we set $\Gamma = \sigma_{\mu\nu} \otimes 1_F \otimes T^a$ and $\Gamma = 1 \otimes 1_F \otimes T^a$ respectively: the longitudinal WT identity reads

$$\langle D_\mu[A]^{ab} \mathcal{J}^{\mu b}(x) \rangle_J = i \langle \bar{\Psi}(x) T^a \eta(x) \rangle_J - i \langle \bar{\eta}(x) T^a \Psi(x) \rangle_J, \quad (3.29)$$

while the transverse WT identity is given by

$$\begin{aligned} & \langle D_\mu[A] \mathcal{J}_\nu(x) \rangle_J - \langle D_\nu[A] \mathcal{J}_\mu(x) \rangle_J \\ = & 2M \langle \bar{\Psi}(x) \sigma_{\mu\nu} T^a \Psi(x) \rangle_J - \langle \bar{\Psi}(x) \sigma_{\mu\nu} T^a \eta(x) \rangle_J - \langle \bar{\eta}(x) \sigma_{\mu\nu} T^a \Psi(x) \rangle_J \\ & - \langle \bar{\Psi}(x) \frac{i}{2} \{ \sigma_{\mu\nu}, \gamma^\rho \} T^a (\vec{\partial}_\rho - \overleftarrow{\partial}_\rho) \Psi(x) \rangle_J \\ & - e \langle \bar{\Psi}(x) \frac{1}{2} \{ \sigma_{\mu\nu}, \gamma_\rho \} \{ T^a, T^b \} \Psi(x) A^{\rho b}(x) \rangle_J, \end{aligned} \quad (3.30)$$

where the covariant derivative $D_\mu[A]$ is defined by

$$D_\mu[A]^{ab} := \delta^{ab} \partial_\mu + e f^{abc} A_\mu^b, \quad (3.31)$$

for our convention $[T^a, T^b] = i f^{abc} T^c$.

4 Chiral WT identity

As a special case of the identities, (3.11) and (3.12), we can derive the chiral WT identities, i.e. WT identity for the axial vector current

$$\mathcal{J}_\mu^5(x) := \bar{\Psi}(x) \gamma_5 \gamma_\mu \Psi(x). \quad (4.1)$$

The choice

$$\Gamma = \gamma_5 \otimes 1_F \otimes 1_C \quad (4.2)$$

in Eq. (3.12) yields the usual chiral WT identity:

$$\partial_\mu \langle \mathcal{J}_5^\mu(x) \rangle_J = 2M \langle \bar{\Psi}(x) \gamma_5 \Psi(x) \rangle_J - \langle \bar{\Psi}(x) \gamma_5 \eta(x) \rangle_J - \langle \bar{\eta}(x) \gamma_5 \Psi(x) \rangle_J, \quad (4.3)$$

up to the quantum anomaly $\mathcal{A}[A_\mu]$ which will be discussed in the next section. Up to the anomalous term, this leads to

$$\begin{aligned} \mathcal{D}_5(x, y, z) &:= \partial_\mu \langle \mathcal{J}_5^\mu(x); \Psi(y); \bar{\Psi}(z) \rangle_c \\ &= \langle \bar{\Psi}(x) \{ \gamma_5, M \} \Psi(x) \Psi(y) \bar{\Psi}(z) \rangle_c \\ &\quad - \langle \Psi(y) \bar{\Psi}(z) \rangle_c \gamma_5 \delta(x - z) - \gamma_5 \langle \Psi(y) \bar{\Psi}(z) \rangle_c \delta(x - y) \\ &\quad + \langle \mathcal{A}[A_\mu]; \Psi(y); \bar{\Psi}(z) \rangle_c. \end{aligned} \quad (4.4)$$

Moreover, choosing

$$\Gamma = \gamma_5 \sigma_{\mu\nu} \otimes 1_F \otimes 1_C \quad (4.5)$$

in Eq. (3.11), we obtain the transverse WT identity for the axial vector vertex:

$$\begin{aligned}\mathcal{R}_{\mu\nu}^5(x)_J &:= -\langle\bar{\Psi}(x)\gamma_5\sigma_{\mu\nu}\eta(x)\rangle_J - \langle\bar{\eta}(x)\sigma_{\mu\nu}\gamma_5\Psi(x)\rangle_J \\ &\quad -\langle\bar{\Psi}(x)\frac{1}{2}\{\sigma_{\mu\nu},\gamma^\rho\}\gamma_5(\vec{\partial}_\rho - \overleftarrow{\partial}_\rho)\Psi(x)\rangle_J \\ &\quad -e\langle\bar{\Psi}(x)\{\sigma_{\mu\nu},\gamma_\rho\}\gamma_5\Psi(x)A^\rho(x)\rangle_J.\end{aligned}\tag{4.6}$$

Then we obtain

$$\begin{aligned}\mathcal{R}_{\mu\nu}^5(x,y,z) &:= \partial_\mu\langle\mathcal{J}_\nu^5(x);\Psi(y);\bar{\Psi}(z)\rangle_c - \partial_\nu\langle\mathcal{J}_\mu^5(x);\Psi(y);\bar{\Psi}(z)\rangle_c \\ &= -\langle\Psi(y)\bar{\Psi}(x)\rangle_c\gamma_5\sigma_{\mu\nu}\delta^D(x-z) - \sigma_{\mu\nu}\gamma_5\langle\Psi(x)\bar{\Psi}(z)\rangle_c\delta^D(x-y) \\ &\quad -\langle\bar{\Psi}(x)\frac{1}{2}\{\sigma_{\mu\nu},\gamma^\rho\}\gamma_5(\vec{\partial}_\rho - \overleftarrow{\partial}_\rho)\Psi(x);\Psi(y);\bar{\Psi}(z)\rangle_c \\ &\quad -e\langle\bar{\Psi}(x)\{\sigma_{\mu\nu},\gamma_\rho\}\gamma_5\Psi(x)A^\rho(x);\Psi(y);\bar{\Psi}(z)\rangle_c.\end{aligned}\tag{4.7}$$

In 1 + 1 dimensional case, the transverse chiral WT identity reduces to the longitudinal (vector) WT identity:

$$\mathcal{R}_{\mu\nu}^5(x)_J = \epsilon_{\mu\nu}\mathcal{D}(x)_J, \quad \mathcal{R}_{\mu\nu}^5(x,y,z) = \epsilon_{\mu\nu}\mathcal{D}(x,y,z),\tag{4.8}$$

since $\mathcal{J}_\mu^5 = \epsilon_{\mu\nu}\mathcal{J}^\nu$, and hence $\partial_\mu\mathcal{J}_\nu^5 - \partial_\nu\mathcal{J}_\mu^5 = \epsilon_{\mu\nu}\partial^\rho\mathcal{J}_\rho$. On the other hand, the longitudinal chiral WT identity reduces to the transverse (vector) WT identity:

$$\mathcal{D}_5(x)_J = \epsilon^{\mu\nu}\mathcal{R}_{\mu\nu}^5(x)_J, \quad \mathcal{D}_5(x,y,z) = \epsilon^{\mu\nu}\mathcal{R}_{\mu\nu}^5(x,y,z),\tag{4.9}$$

since $\partial^\mu\mathcal{J}_\mu^5 = \epsilon_{\mu\nu}\partial^\mu\mathcal{J}^\nu = \frac{1}{2}\epsilon_{\mu\nu}(\partial^\mu\mathcal{J}^\nu - \partial^\nu\mathcal{J}^\mu)$. The extension to the non-Abelian case is straightforward for the axial vector current

$$\mathcal{J}_\mu^{5a}(x) := \bar{\Psi}(x)\gamma_5\gamma_\mu T^a\Psi(x),\tag{4.10}$$

as in the previous section.

5 Anomaly

5.1 general case

In the derivation of the WT identity so far, we have not taken into account the anomaly. Now we examine a possibility of the existence of the anomaly based on the path integral formalism. The method of deriving the anomaly in the path integral formalism is well known as Fujikawa's method [14]. The anomaly $\mathcal{A}[A](x)$ comes from the Jacobian factor $J[A]$ which is accompanied by the change of variable in the path integral measure: ⁴

$$\mathcal{D}\bar{\Psi}'\mathcal{D}\Psi' = \mathcal{D}\bar{\Psi}\mathcal{D}\Psi J[A], \quad J[A] = \exp\left[-\int d^Dx \alpha(x)\mathcal{A}[A](x)\right].\tag{5.1}$$

⁴ The Fujikawa method is applied only in the Euclidean space. Hence the anomaly obtained in this method is the Euclidean version of the corresponding anomaly in Minkowski space-time, although we write the anomaly as if we obtained it in Minkowski space-time in what follows.

Actually, all the WT identities derived so far in this paper can also be derived by appropriate change of variable together with the possible anomaly, as shown in the following. In the calculation of the anomaly, the gauge-boson field is identified as the external field. Therefore, it is enough to pay attention only to the fermionic part \mathcal{L}_F . Perform the following change of variable: ⁵

$$\begin{aligned}\Psi(x) &\rightarrow \Psi'(x) := e^{i\Omega\alpha(x)}\Psi(x), \\ \bar{\Psi}(x) &\rightarrow \bar{\Psi}'(x) := \bar{\Psi}(x)e^{i\tilde{\Omega}\alpha(x)},\end{aligned}\tag{5.2}$$

where Ω and $\tilde{\Omega}$ are global quantities which are similar to Γ used in the previous derivation of the WT identity. If we put $\tilde{\Omega} = \pm\Omega$, the change of the fermionic action $S_F = \int d^D x \mathcal{L}_F$ under the transformation (5.2) is written (for small parameter α) as

$$\begin{aligned}\delta_\Omega S_F &= \int d^D x \left\{ \bar{\Psi} \frac{1}{2} [i\gamma^\mu, i\Omega\partial_\mu\alpha]_\mp \Psi \right. \\ &\quad + \bar{\Psi} \frac{1}{2} [\gamma^\mu, i\Omega\alpha]_\pm \partial_\mu \Psi - \partial_\mu \bar{\Psi} \frac{1}{2} [\gamma^\mu, i\Omega\alpha]_\pm \Psi \\ &\quad + e \bar{\Psi} [\gamma^\mu, i\Omega\alpha]_\pm \Psi - \bar{\Psi} [M, i\Omega\alpha]_\pm \Psi \\ &\quad \left. + \bar{\eta} i\Omega\alpha \Psi + \bar{\Psi} i\tilde{\Omega}\alpha\eta \right\} + O(\alpha^2),\end{aligned}\tag{5.3}$$

where we have employed the notation for the anticommutation and the commutation relations: $[A, B]_+ := \{A, B\}$ and $[A, B]_- := [A, B]$. The theory is invariant under the change of variable:

$$0 = \left\langle \frac{\delta_\Omega S}{\delta\alpha^a(x)} \right|_{\alpha=0} \rangle_J.\tag{5.4}$$

Then, after performing the integration by parts, we obtain

$$\begin{aligned}&\partial_\rho \langle \bar{\Psi}(x) \frac{1}{2} [i\gamma^\rho, i\Omega^a]_\mp \Psi(x) \rangle_J \\ &= -\langle \bar{\Psi}(x) [M, i\Omega^a]_\pm \Psi(x) \rangle_J + \langle \bar{\eta}(x) i\Omega^a \Psi(x) \rangle_J \pm \langle \bar{\Psi}(x) i\Omega^a \eta(x) \rangle_J \\ &\quad + \langle \bar{\Psi}(x) \frac{1}{2} [\gamma^\rho, i\Omega^a]_\pm (\vec{\partial}_\rho - \overleftarrow{\partial}_\rho) \Psi(x) \rangle_J \\ &\quad + e \langle \bar{\Psi}(x) [\gamma^\rho A_\rho(x), i\Omega^a]_\pm \Psi(x) \rangle_J + \langle \mathcal{A}_\Omega[A](x) \rangle_J,\end{aligned}\tag{5.5}$$

where we have set $\alpha(x) = \alpha^a(x)T^a$ ($a = 1, \dots, \dim G$) and defined $\Omega^a := \Omega T^a$. In the Abelian case, we can omit the index and put $T^a \equiv 1$. Here $\mathcal{A}_\Omega[A](x)$ denotes symbolically the anomaly accompanied by the above transformation (5.2). Up to the anomaly, (5.5) is equivalent to a pair of Eq. (3.11) and (3.12).

First, we consider the case $\tilde{\Omega} = +\Omega$ which corresponds to the upper signature in Eq. (5.5). In this case,

$$\Omega = \gamma_5 \otimes 1_F \otimes 1_C\tag{5.6}$$

⁵ In the Euclidean space, Ψ and $\bar{\Psi}$ are independent Grassmann variables, in sharp contrast with the Minkowski space-time where $\bar{\Psi} = \Psi^\dagger \gamma^0$. Therefore, the independent change of variable is allowed in Euclidean space.

reproduces the chiral WT identity. Actually, the change of variable (5.2) reproduces correctly the chiral anomaly as shown by Fujikawa [14]. Especially, for Abelian gauge theory in $D = 4$ dimensions [2]

$$\mathcal{A}_\Omega[A] = \frac{-i}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}. \quad (5.7)$$

The transverse WT identity is reproduced for

$$\Omega = \epsilon^{\mu\nu} \sigma_{\mu\nu} \otimes 1_F \otimes 1_C. \quad (5.8)$$

In this case, the change of variable (5.2) agrees with the local Lorentz transformation

$$\Omega\alpha(x) = \epsilon^{\mu\nu} \sigma_{\mu\nu} \alpha(x). \quad (5.9)$$

Usually it is understood that there is no Lorentz anomaly in the theory of the Dirac fermion in the flat space-time, in contrast with the parity-breaking Weyl fermion in the presence of non-trivial background gravitational field in $D \geq 4$ dimensions [15]. Therefore we conclude that there is no anomaly for the transverse WT identity and hence there is no correction to the expression in $D \geq 4$ dimensions for the transverse WT identity (3.17) derived in the previous method.

On the other hand, the case of $\tilde{\Omega} = -\Omega$ corresponds to the lower signature in Eq. (5.5). The longitudinal (vector) WT identity is obtained from

$$\Omega = 1_S \otimes 1_F \otimes 1_C. \quad (5.10)$$

Finally, the transverse chiral WT identity corresponds to

$$\Omega = \epsilon^{\mu\nu} \gamma_5 \sigma_{\mu\nu} \otimes 1_F \otimes 1_C. \quad (5.11)$$

For this choice, as far as we know, there are no references which have investigated the corresponding anomaly. As we do not need the detail of this case in the following part of this article and its investigation is off the main stream of this article, we do not pursue this issue any longer. However, the 1+1 dimensional case is simple and discussed below. The higher dimensional case deserves further studies.

5.2 1+1 dimensions

It should be remarked that in 1+1 dimension the situation is somewhat different from the higher dimensional case. The transverse chiral WT identity is essentially the same as the (longitudinal) WT identity, since $\Omega = \epsilon^{\mu\nu} \gamma_5 \sigma_{\mu\nu} = \epsilon^{\mu\nu} \gamma_5 \epsilon_{\mu\nu} \gamma_5 = 21_S$. They are both anomaly free, if we impose the gauge invariance. Similarly, as $\Omega = \epsilon^{\mu\nu} \sigma_{\mu\nu} = 2\gamma_5$, the transverse WT identity gives the equivalent information to the chiral (longitudinal) WT identity in 1+1 dimensions. In this case, the information of the anomaly can be also translated into the rotation of the vector current, since the equation

$$\partial^\mu \mathcal{J}_\mu^5 = \mathcal{A}_2, \quad \mathcal{A}_2 := -\frac{e}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu}, \quad (5.12)$$

is equivalent to

$$\partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu = -\frac{e}{\pi} F_{\mu\nu}. \quad (5.13)$$

Nevertheless we show that there is no modification in the chiral WT identity (4.6) due to the existence of the chiral anomaly \mathcal{A}_2 . This is because the chiral anomaly is promoted to the modification of the gauge-boson propagator as dynamical mass generation (by radiative quantum correction) $\mu := e/\sqrt{\pi}$ for the gauge-boson A_μ . In fact, for the vacuum polarization tensor in 1+1 dimensions of the form:

$$\Pi_{\mu\nu}(k) = \frac{e^2}{\pi} (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}), \quad (5.14)$$

Eq. (3.2) is modified as follows.

$$\begin{aligned} & -\frac{1}{e} [\partial_\nu D_{\mu\rho}^{-1}(\partial) - \partial_\mu D_{\nu\rho}^{-1}(\partial)] A^\rho \\ &= -\frac{1}{e} (\partial^2 + \frac{e^2}{\pi}) (\partial_\mu g_{\nu\rho} - \partial_\nu g_{\mu\rho}) A^\rho \\ &= -\frac{1}{e} [\partial_\nu D_{\mu\rho}^{(0)-1}(\partial) - \partial_\mu D_{\nu\rho}^{(0)-1}(\partial)] A^\rho - \frac{e}{\pi} F_{\mu\nu}. \end{aligned} \quad (5.15)$$

Indeed, the additional term in this equation agrees with the right-hand-side of Eq. (5.13).

6 Application to the SD equation

In what follows we restrict our study to the Abelian gauge theory.

In momentum representation, the SD equation for the full fermion propagator $S(p)$ is given by

$$S_0^{-1}(p)S(p) = 1 + ie^2 \int \frac{d^D k}{(2\pi)^D} \gamma^\mu D_{\mu\nu}(k) S(p-k) \Gamma_\nu(p-k, p) S(p), \quad (6.1)$$

where $D_{\mu\nu}(k)$ is the full gauge-boson propagator, $\Gamma_\nu(p-k, p)$ the full vertex function and S_0 the bare fermion propagator:

$$S_0(p) := \frac{1}{\hat{p} - m}, \quad (\hat{p} := \gamma^\mu p_\mu). \quad (6.2)$$

Now we introduce more general gauge fixing than the usual one, which is called the nonlocal gauge-fixing, see e.g. [16]. In the configuration space, the gauge fixing term in the nonlocal gauge is given by

$$\mathcal{L}_{GF} = -\frac{1}{2} F[A(x)] \int d^D y \frac{1}{\xi(x-y)} F[A(y)], \quad (6.3)$$

for the covariant gauge $F[A] = \partial^\mu A_\mu$. In momentum representation, the gauge-fixing parameter ξ gets momentum-dependent, i.e. ξ becomes a function of the momentum: $\xi = \xi(k^2)$. Here it should be noted that $\xi^{-1}(k^2)$ is the Fourier transform of $\xi^{-1}(x)$,

$$\xi^{-1}(x) = \int \frac{d^D k}{(2\pi)^D} e^{ikx} \xi^{-1}(k^2), \quad \xi^{-1}(k^2) = \int d^D x e^{-ikx} \xi^{-1}(x), \quad (6.4)$$

while $\xi(k^2)$ is not the Fourier transform of $\xi(x)$, see ref. [16]. If $\xi(k^2)$ does not have the momentum-dependence, i.e., $\xi(k^2) \rightarrow \xi$, then $\xi^{-1}(x-y) \rightarrow \delta(x-y)\xi^{-1}$ and the nonlocal gauge-fixing term reduces to the usual gauge-fixing term:

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(F[A(x)])^2. \quad (6.5)$$

Hence, in the nonlocal gauge, the bare gauge-boson propagator $D_{\mu\nu}^{(0)}(k)$ is given by

$$D_{\mu\nu}^{(0)-1}(k) = k^2 g_{\mu\nu} - k_\mu k_\nu + \xi(k)^{-1} k_\mu k_\nu. \quad (6.6)$$

On the other hand, the SD equation for the full gauge-boson propagator is given by

$$\begin{aligned} D_{\mu\nu}^{-1}(k) &= D_{\mu\nu}^{(0)-1}(k) - \Pi_{\mu\nu}(k), \\ \Pi_{\mu\nu}(k) &:= e^2 \int \frac{d^D p}{(2\pi)^D} \text{tr}[\gamma_\mu S(p) \Gamma_\nu(p, p-k) S(p-k)]. \end{aligned} \quad (6.7)$$

In the gauge theory, the vacuum polarization tensor should have the transverse form:

$$\Pi_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi(k), \quad (6.8)$$

as far as the gauge invariance is preserved. Hence the full gauge-boson propagator is of the form

$$\begin{aligned} D_{\mu\nu}(k) &= D_T(k) \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \frac{\xi(k)}{k^2} \frac{k_\mu k_\nu}{k^2} = D_{\mu\nu}^T(k) + D_{\mu\nu}^L(k), \\ D_T(k) &:= \frac{1}{k^2 - \Pi(k)}. \end{aligned} \quad (6.9)$$

In the following we wish to give comments on how to choose the ansatz for the vertex function Γ_μ in solving the SD equation for the fermion propagator.

6.1 decomposition of the vertex function

To solve the SD equation for the fermion propagator, we do not need to know the explicit form of the vertex function Γ_μ itself, although much effort has been devoted to find an ansatz for the full vertex function [9, 10, 8, 11]. It is enough to specify the divergence $\partial_\mu \Gamma^\mu$ and the rotation $\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu$ of the vertex, not the vertex itself, since the vertex function appears only in this combination in the integrand of the SD equation for the fermion propagator. This can be observed as follows. In Eq. (6.1),

$$\begin{aligned} & D_{\mu\nu}(k) \tilde{\Gamma}^\nu(q, p) \\ &= D_{\mu\nu}^L(k) \tilde{\Gamma}^\nu(q, p) + D_{\mu\nu}^T(k) \tilde{\Gamma}^\nu(q, p) \\ &= \frac{\xi(k^2)}{k^2} \frac{k_\mu}{k^2} [k_\nu \tilde{\Gamma}^\nu(q, p)] + D_T(k) \frac{k^\nu}{k^2} [k_\nu \tilde{\Gamma}_\mu(q, p) - k_\mu \tilde{\Gamma}_\nu(q, p)], \end{aligned} \quad (6.10)$$

where we have defined $q := p - k$ and

$$\tilde{\Gamma}_\nu(q, p) := S(q)\Gamma_\nu(q, p)S(p). \quad (6.11)$$

The divergence $k_\nu \tilde{\Gamma}^\nu(q, p)$ is known exactly from the usual WT identity (2.18). For the rotation, $k_\nu \tilde{\Gamma}_\mu(q, p) - k_\mu \tilde{\Gamma}_\nu(q, p)$, the transverse WT identity (3.17) gives a clue to find the correct expression. Moreover, the combination $\tilde{\Gamma}_\mu := S(p)\Gamma_\mu(p, q)S(q)$ is more convenient rather than $\Gamma_\mu(p, q)$ to specifying the vertex as shown in the following subsections.

The decomposition in (6.10) can be written as

$$D_{\mu\nu}(k)\tilde{\Gamma}^\nu(q, p) = \frac{\xi(k^2)}{k^2}\tilde{\Gamma}_\mu{}^L(q, p) + D_T(k)\tilde{\Gamma}_\mu{}^T(q, p). \quad (6.12)$$

Therefore, the specification of the transverse vertex to the form

$$\Gamma_\mu{}^T(q, p) = \frac{k^\nu}{k^2}[k_\nu\Gamma_\mu(q, p) - k_\mu\Gamma_\nu(q, p)] = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Gamma^\nu(q, p) \quad (6.13)$$

is *allowed only in the integrand of the SD equation for the fermion propagator*, as well as

$$\Gamma_\mu{}^L(q, p) = \frac{k^\mu}{k^2}[k_\nu\Gamma^\nu(q, p)] = \frac{k^\mu}{k^2}[S^{-1}(q) - S^{-1}(p)]. \quad (6.14)$$

Note that the factor $1/k^2$ comes from the gauge-boson propagator in the integrand, see [17]

6.2 gauge choice and LK transformation

If we know a set of solutions (for the full gauge boson propagator, the full fermion propagator and the full vertex function) in a *single* gauge, the solutions in other gauges are obtained through the Landau-Khalatnikov (LK) transformation [18]:

$$\begin{aligned} D'_{\mu\nu}(x) &= D_{\mu\nu}(x) + \partial_\mu\partial_\nu f(x), \\ S'(x, y) &= e^{e^2[f(o)-f(x-y)]}S(x, y), \\ \mathcal{V}'_\nu(x, y, z) &= e^{e^2[f(o)-f(x-y)]}\mathcal{V}_\nu(x, y, z) \\ &\quad + S(x, y)e^{e^2[f(o)-f(x-y)]}\partial_\nu^z[f(x-z) - f(z-y)], \end{aligned} \quad (6.15)$$

where

$$\begin{aligned} D_{\mu\nu}(x, y) &= \langle 0|T[A_\mu(x)A_\nu(y)]|0\rangle, \\ S(x, y) &= \langle 0|T[\Psi(x)\bar{\Psi}(y)]|0\rangle, \\ \mathcal{V}_\nu(x, y, z) &= \langle 0|T[\Psi(x)\bar{\Psi}(y)A_\nu(z)]|0\rangle. \end{aligned} \quad (6.16)$$

Here the function $f(x)$ allows a nonlocal gauge fixing. Actually the SD equation is form-invariant under the LK transformation. This can be easily shown in the coordinate space where the SD equation has the following form:

$$(i\hat{\partial} - m)S(x, y) = \delta^D(x - y) + ie^2\gamma^\mu\langle\Psi(x)\bar{\Psi}(y)A_\mu(x)\rangle, \quad (6.17)$$

and

$$\begin{aligned}
D_{\mu\nu}^{-1}(x, z) &= D_{\mu\nu}^{(0)-1}(x, z) - \Pi_{\mu\nu}(x, z), \\
\Pi_{\mu\nu}(x, z) &= (g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\Pi(x - z) \\
&= ie^2 \int d^D z_1 d^D z_2 \text{tr}[\gamma_\mu S(x, z_1) \Gamma_\nu(z_1, z_2; z) S(z_2, x)], \quad (6.18)
\end{aligned}$$

where

$$\langle \Psi(x) \bar{\Psi}(y) A_\mu(z) \rangle = \int d^D x' d^D y' d^D z' S(x, x') \Gamma_\nu(x', y'; z') S(y', y) D_{\mu\nu}(z', z). \quad (6.19)$$

Under the LK transformation, the divergence $\partial_\mu \tilde{\Gamma}^\mu$ and the rotation $\partial_\mu \tilde{\Gamma}_\nu - \partial_\nu \tilde{\Gamma}_\mu$ formed from $\tilde{\Gamma}^\mu$, not from Γ^μ , obey the following simple transformation law which is the same as that of the full fermion propagator (6.15):

$$\begin{aligned}
\mathcal{D}'(x, y, z) &= e^{e^2[f(o)-f(x-y)]} \mathcal{D}(x, y, z), \\
\mathcal{R}'_{\mu\nu}(x, y, z) &= e^{e^2[f(o)-f(x-y)]} \mathcal{R}_{\mu\nu}(x, y, z). \quad (6.20)
\end{aligned}$$

So the problem reduces to find a set of solutions in a single gauge.

6.3 non-linearity

As is usually done, if one starts from the SD equation for the inverse fermion propagator $S^{-1}(p)$:

$$S^{-1}(p) = S_0^{-1}(p) + ie^2 \int \frac{d^D k}{(2\pi)^D} \gamma^\mu D_{\mu\nu}(k) S(p - k) \Gamma_\nu(p - k, p), \quad (6.21)$$

and decomposes it into a pair of integral equations for A and B using

$$S^{-1}(p) = A(p)\hat{p} - B(p), \quad (6.22)$$

the SD equations become a pair of non-linear integral equations for A and B . Therefore, when one wishes to solve the equation analytically, some type of linearization is indispensable in most cases. The linearization approximation is sufficient to study some restricted kinds of problems, e.g. the critical behavior in the neighborhood of the critical point at which $B(p) \equiv 0$. However, the solution far from the critical point or the solution in the time-like region [5, 19, 20] can be studied only through the non-linear equation. If we could express the vertex function $\tilde{\Gamma}_\nu(q, p) = S(q)\Gamma_\nu(q, p)S(p)$ as a functional of the fermion propagator S and solve the SD equation in terms of $S(p)$, not $S^{-1}(p)$, we could obtain the solution which includes the non-linear effect without the linearization and/or the decomposition in the following sense. If we start from the SD equation (6.1) for S into which

$$S(p) = \frac{\hat{p}X(p) + Y(p)}{p^2}, \quad (6.23)$$

is substituted, a pair of integral equations for X and Y is obtained. In general, although X and Y couple each other, the equations are still linear in each variable, X or Y . This is not the case for a pair of equations for A and B obtained from the decomposition of the SD equation for $S^{-1}(p)$. It is much easier to obtain the solution, X and Y , for such equations which include the non-linear effect through

$$X(p) := \frac{A(p)p^2}{A^2(p)p^2 + B^2(p)}, \quad Y(p) := \frac{B(p)p^2}{A^2(p)p^2 + B^2(p)}. \quad (6.24)$$

Note that, for B small,

$$X(p) \cong \frac{1}{A(p)}, \quad Y(p) := \frac{B(p)}{A^2(p)}. \quad (6.25)$$

while, for B large

$$X(p) := \frac{A(p)p^2}{B^2(p)}, \quad Y(p) := \frac{p^2}{B(p)}. \quad (6.26)$$

The wavefunction renormalization $A(p)$ and the mass function $M(p)$ are obtained from

$$M(p) := \frac{B(p)}{A(p)} = \frac{Y(p)}{X(p)}, \quad Z(p) := \frac{1}{A(p)} = X(p) \left(1 + \frac{M^2(p)}{p^2} \right). \quad (6.27)$$

In the exact sense, it is impossible to realize such a situation in general. However, we find, in 1+1 dimensional case, such a simple situation is realized and we can solve the SD equation without any approximation. In this case the SD equation (6.28) is linear in S and can be reduced to a decoupled pair of integral equations for X and Y .

6.4 exactly soluble truncated SD equation

At first glance, the right-hand-side of SD equation (6.1) seems to be quadratic in S . This observation is not right, because at least the longitudinal WT identity for the vertex function (2.15) gives a linear part. If the transverse part for the vertex is also linear in S , the SD equation might be solved exactly. In order to examine such a possibility, we try to use the truncated form of the transverse WT identity up to the same form as the (3.27). Substituting the longitudinal WT identity (2.18) and the truncated transverse WT identity (3.27) into (6.1) by way of (6.10), we get the SD equation for the fermion propagator (in the chiral limit $M = 0$):

$$\hat{p}S(p) = 1 + ie^2 \int \frac{d^D k}{(2\pi)^D} S(p-k) \hat{k} \left[(D-1) \frac{D_T(k)}{k^2} - \frac{\xi(k^2)}{k^4} \right], \quad (6.28)$$

where we have used $\gamma_\mu \sigma_{\nu\mu} = i(1-D)\gamma_\nu$ and dropped the identically vanishing integral. This SD equation is exact only in 1+1 space-time dimensions. The solution is easily

found by moving to the coordinate space, since the Fourier transformation changes the convolution in momentum space into the simple product in the coordinate space:

$$i\hat{\partial}S(x) = \delta^D(x) - ie^2 S(x) \int \frac{d^D k}{(2\pi)^D} \hat{k} \left[\frac{D_T(k)}{k^2} - \frac{\xi(k^2)}{k^4} \right] e^{-ik \cdot x}. \quad (6.29)$$

Once D_T is given and does not include S ,⁶ this equation can be solved exactly and the solution is given by

$$S(x) = S_0(x) \exp \left\{ -ie^2 \int \frac{d^D k}{(2\pi)^D} \left[\frac{D_T(k)}{k^2} - \frac{\xi(k^2)}{k^4} \right] (e^{-ik \cdot x} - 1) \right\}, \quad (6.30)$$

where we have assumed the translational invariance $S(x, y) = S(x - y, 0) := S(x - y)$. It is obvious that this solution is consistent with the LK transformation Eq. (6.20).

The above truncation for the vertex is consistent with the multiplicative renormalizability and the LK transformation. The detailed comparison with the decomposition of Ball-Chiu [12] will be given in a subsequent paper [21], together with the solution of the SD equation in $D=2+1$ and $3+1$ dimensions.

7 1+1 dimensional case

In $D = 1 + 1$ dimensions, by virtue of the simple transverse WT identity (3.27) and the usual longitudinal WT identity (2.18), we can write down the exact and closed integral equation for the SD equation for the fermion propagator $S(x)$ in the massless bare fermion limit $M = 0$, which can be solved without approximation. From the consideration in the previous section, the exact full fermion propagator of massless QED₂ is given by

$$\begin{aligned} S(x) &= S_0(x) \exp \left\{ -ie^2 \int \frac{d^2 k}{(2\pi)^2} \left[\frac{D_T(k)}{k^2} - \frac{\xi(k^2)}{k^4} \right] (e^{-ik \cdot x} - 1) \right\}, \\ D_T(k) &= \frac{1}{k^2 - \frac{e^2}{\pi}}, \end{aligned} \quad (7.1)$$

where the bare fermion propagator is given by

$$S_0(p) := \frac{1}{\hat{p}}, \quad S_0(x) = \int \frac{d^2 p}{(2\pi)^2} e^{ip \cdot x} S_0(p) = \frac{1}{4\pi} \frac{\hat{x}}{x^2}. \quad (7.2)$$

In the $\xi = 0$ gauge, this result coincides with Schwinger's result [23]. Obviously this solution obeys the LK transformation. Therefore this solution gives the exact solution in the arbitrary gauge ξ . In fact, it turns out that this solution agrees with the exact solution obtained in the path integral formalism. The exact solution of this type is obtained for more general models in 1+1 dimension, e.g. the gauged Thirring model [24]. This issue will be discussed separately [21].

⁶ The quenched approximation is the simplest case where $D_T(k) = 1/k^2$.

In massless QED₂ (massless Schwinger model), the gauge field acquires a mass $e/\sqrt{\pi}$ due to radiative correction which can also be interpreted as a consequence of anomaly. It is usually claimed that the exact solution (7.1) has no extra pole at $p \neq 0$ and hence the fermion remains massless. Nevertheless, we can show that the exact solution does not exclude the dynamical fermion mass generation in the chiral limit $M = 0$. This can be shown by solving the Euclidean version of the SD equation in momentum space [21].

8 Conclusion and discussion

In this paper, we rederived the transverse WT and the chiral WT identities based on the path integral formalism and examined the possible existence of anomaly for such a new type of WT identities. In the framework of the SD equation, we have proposed the strategy in which the transverse WT identity as well as the usual longitudinal WT identity is used to find the appropriate ansatz for the vertex function in writing down the SD equation for the fermion propagator. Especially, in $D = 2$ dimensions, we have shown that this strategy can be performed without any approximation and leads to the exact SD equation for the fermion propagator which can be exactly solved.

It is interesting to try to solve the SD equation for the fermion propagator in $D > 2$ dimensions under an ansatz for the vertex function which is guided by the transverse WT identity (as well as the longitudinal WT identity). Such a preliminary attempt has been done already in [25] by truncating the transverse WT identity up to the nontrivial order in strong coupling QED in 3+1 dimensions. More detailed investigation based on the scheme proposed in this paper will be given in a subsequent work [26].

Acknowledgments

The author would like to thank Dr. Ian J.R. Aitchison for kind hospitality in Oxford. He is also very grateful to Prof. Yashushi Takahashi who brought his attention to the transverse WT identity in the very early stage of this work [25] in 1988. This work is supported in part by the Japan Society for the Promotion of Science and the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (No.07640377).

References

- [1] J.C. Ward, Phys. Rev. 78 (1950) 1824.
Y. Takahashi, Nuovo Cimento 6 (1957) 370.
- [2] S. Adler, Phys. Rev. 177 (1969) 2426.
J.S. Bell and R. Jackiw, Nuovo Cimento A 60 (1969) 47. S. Adler and W.A. Bardeen, Phys. Rev. 182 (1969) 1517.
- [3] F.J. Dyson, Phys. Rev. 75 (1949) 1736.
J. Schwinger, Proc. Nat. Acad. Sc. 37 (1951) 452, 455.
- [4] T. Maskawa and H. Nakajima, Prog. Theor. Phys. 52 (1974) 1326.
- [5] R. Fukuda and T. Kugo, Nucl. Phys. B 117 (1976) 250.
- [6] V.A. Miransky, Nuovo Cimento 90 A (1985) 149.
P.I. Fomin, V.P. Gusynin, V.A. Miransky and Yu.A. Sitenko, La rivista del Nuovo Cimento 6 (1983) 1.
- [7] J.B. Kogut, E. Dagotto and A. Kocic, Phys. Rev. Lett. 60 (1988) 772; 61 (1988) 2416. Nucl. Phys. B 317 (1989) 253; *ibid.* 271.
- [8] C.J. Burden, J. Praschifka and C.D. Roberts, Phys. Rev. D 46 (1992) 2695.
C.J. Burden, and C.D. Roberts, Phys. Rev. D 47 (1993) 5581.
Z. Dong, H.J. Munczek and C.D. Roberts, Phys. Lett. B 333 (1994) 536.
- [9] D. Atkinson, J.C.R. Bloch, V.P. Gusynin, M.R. Pennington and M. Reenders, Phys. Lett. B 329 (1994) 117.
- [10] A. Kizilersu, M. Reenders and M.R. Pennington, Phys. Rev. D 52 (1995) 1242.
D.C. Curtis and M.R. Pennington, Phys. Rev. D 42 (1990) 4165.
- [11] A. Bashir and M.R. Pennington, hep-ph/9510436, Phys. Rev. D (1996).
A. Bashir and M.R. Pennington, Phys. Rev. D 50 (1994) 7679.
- [12] J.S. Ball and T.-W. Chiu, Phys. Rev. D 22 (1980) 2542.
- [13] Y. Takahashi, in 'Quantum Field Theory,' ed. by F. Mancini (Elsevier Science Publishers, 1986).

- [14] K. Fujikawa, Phys. Rev. Lett. 42 (1979) 1195. Phys. Rev. D 21 (1980) 2848.
- [15] K. Fujikawa, Nucl. Phys. B 226 (1983) 437.
 L. Alvarez-Gaume and P. Ginsparg, Ann. Phys. (NY) 161 (1985) 423.
 L. Alvarez-Gaume and E. Witten, Nucl. Phys. B 234 (1984) 269.
 W.A. Bardeen and B. Zumino, Nucl. Phys. B 244 (1984) 421.
- [16] K.-I. Kondo and P. Maris, Phys. Rev. D. 52 (1995) 1212.
- [17] K.-I. Kondo, Intern. J. Mod. Phys. A 7 (1992) 7239.
- [18] L.D. Landau and I.M. Khalatnikov, Zh. Eksp. Teor. Fiz. 29 (1956) 89 [Sov. Phys. JETP 2 (1956) 69]. B. Zumino, J. Math. Phys. 1 (1960) 1.
- [19] D. Atkinson and D.W.E. Blatt, Nucl.Phys. B 151 (1979) 342.
- [20] P. Maris, *Nonperturbative analysis of the fermion propagator: complex singularities and dynamical mass generation*, Ph.D. Thesis, Univ. of Groningen, October 1993.
- [21] K.-I. Kondo, *An Exact Solution of Gauged Thirring Model in Two Dimensions*, Chiba Univ. Preprint, CHIBA-EP-95, hep-th/960????.
- [22] K. Stam, J. Phys. G: Nucl. Phys. 9 (1983) L229.
 R. Delbourgo and G. Thompson, J. Phys. G: Nucl. Phys. 8 (1982) L185.
 R. Delbourgo and T.J. Shepherd, J. Phys. G: Nucl. Phys. 4 (1978) L197.
- [23] J. Schwinger, Phys. Rev. 128 (1962) 2425.
- [24] K.-I. Kondo, *The gauged Thirring model*, hep-th/9603151,
 Chiba Univ. Preprint, CHIBA-EP-93.
- [25] K.-I. Kondo and Y. Kikukawa, *Spontaneous Breaking Of Chiral Symmetry In Quantum Electrodynamics: Beyond The Ladder Approximation (I)*, Nagoya Univ. Preprint, DPNU-88-20-REV, unpublished.
 H. Mino, in the Proceedings of the 1988 International Workshop on New Trends in Strong Coupling Gauge Theories, Nagoya, Aug. 24-27, eds. by M. Bando, T. Muta and K. Yamawaki (World Scientific Co., Singapore, 1989).
- [26] K.-I. Kondo, in preparation.